

Infinite dimensional Hilbert tensors on spaces of analytic functions

Yisheng Song* and Liquan Qi †

November 9, 2016

Abstract

In this paper, the m -order infinite dimensional Hilbert tensor (hypermatrix) is introduced to define an $(m-1)$ -homogeneous operator on the spaces of analytic functions, which is called Hilbert tensor operator. The boundedness of Hilbert tensor operator is presented on Bergman spaces A^p ($p > 2(m-1)$). On the base of the boundedness, two positively homogeneous operators are introduced to the spaces of analytic functions, and hence the upper bounds of norm of such two operators are found on Bergman spaces A^p ($p > 2(m-1)$). In particular, the norms of such two operators on Bergman spaces $A^{4(m-1)}$ are smaller than or equal to π and $\pi^{\frac{1}{m-1}}$, respectively.

Key words: Hilbert tensor, Analytic function, Upper bound, Bergman space, Gamma function.

AMS subject classifications (2010): 30H10, 30H20, 30H05, 30C10, 47H15, 47H12, 34B10, 47A52, 47J10, 47H09, 15A48, 47H07.

1 Introduction

The Hilbert matrix H is a matrix with entries H_{ij} being the unit fractions for nonnegative integers i, j , i.e.,

$$H_{ij} = \frac{1}{i+j+1}, \quad i, j = 0, 1, 2, \dots$$

*School of Mathematics and Information Science and Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Henan Normal University, XinXiang HeNan, P.R. China, 453007. Email: songyisheng@htu.cn. This author's work was supported by the National Natural Science Foundation of P.R. China (Grant No. 11571095, 11601134)

†Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. Email: liquan.qi@polyu.edu.hk. This author's work was supported by the National Natural Science Foundation of P.R. China (Grant No. 11571095) and by the Hong Kong Research Grant Council (Grant No. PolyU 501212, 501913, 15302114 and 15300715).

which was introduced by Hilbert [1]. Let $i, j = 0, 1, 2, \dots, n$. Then such an n -dimensional Hilbert matrix is a compact linear operator on finite dimensional space \mathbb{R}^n . The properties of n -dimensional Hilbert matrix had been studied by Frazer [2] and Taussky [3]. An infinite dimensional Hilbert matrix H may be regarded as a bounded linear operator from the sequence space l^2 into itself, but not compact operator (Choi [4]) and Ingham [5]). Magnus [6] and Kato [7] showed the spectral properties of such a class of matrices. The infinite dimensional Hilbert matrix H induces an operator defined on the sequence space l^p ($1 \leq p$), for $x = (x_k)_{k=0}^\infty \in l^p$,

$$H(x) = \left(\sum_{j=0}^{+\infty} \frac{x_j}{i+j+1} \right)_{i=0}^\infty. \quad (1.1)$$

If $1 < p < +\infty$, the well-known Hilbert inequality ([37]) implies that H is an operator on l^p and its operator norm $\|H\| = \sup_{\|x\|_{l^p}=1} \|H(x)\|_{l^p}$ is the following:

$$\|H\| = \frac{\pi}{\sin(\frac{\pi}{p})}, \quad 1 < p < +\infty. \quad (1.2)$$

On the other hand, the infinite dimensional Hilbert matrix H also induces a bounded operator on the spaces of analytic functions defined by

$$H(f)(z) = \sum_{i=0}^{+\infty} \left(\sum_{j=0}^{+\infty} \frac{a_j}{i+j+1} \right) z^i, \quad (1.3)$$

for all analytic functions $f(z) = \sum_{k=0}^{+\infty} a_k z^k$ with the convergent coefficients $\sum_{j=0}^{+\infty} \frac{a_j}{i+j+1}$ for each i . In Hardy spaces H^p , Diamantopoulos and Siskakis [8] proved that H is bounded for $p > 1$ and found an upper boundedness of its operator norm. In 2004, Diamantopoulos [9] showed that H is bounded on Bergman spaces for $p > 2$ and obtained the upper boundedness of its operator norm. Aleman, Montes-Rodriguez, Sarason [10] studied the eigenfunctions of Hilbert matrix operator on Hardy space H^p ($p > 1$).

As a natural extension of a Hilbert matrix, the entries of an m -order infinite dimensional Hilbert tensor (hypermatrix) $\mathcal{H} = (\mathcal{H}_{i_1 i_2 \dots i_m})$ are defined by

$$\mathcal{H}_{i_1 i_2 \dots i_m} = \frac{1}{i_1 + i_2 + \dots + i_m + 1}, \quad i_k = 0, 1, 2, \dots, \quad k = 1, 2, \dots, m.$$

Each entry of \mathcal{H} is derived from the integral

$$\mathcal{H}_{i_1 i_2 \dots i_m} = \int_0^1 t^{i_1 + i_2 + \dots + i_m} dt. \quad (1.4)$$

Clearly, \mathcal{H} are positive ($\mathcal{H}_{i_1 i_2 \dots i_m} > 0$) and symmetric ($\mathcal{H}_{i_1 i_2 \dots i_m}$ are invariant for any permutation of the indices), and \mathcal{H} is a Hankel tensor with $v = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$ (Qi [11]).

Song and Qi [12] studied infinite and finite dimensional Hilbert tensors, and showed that \mathcal{H} defines a bounded and positively $(m-1)$ -homogeneous operator from l^1 into l^p ($1 < p < \infty$), and found the upper boundedness of corresponding positively homogeneous operator norm.

A real m -order n -dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \dots i_m})$ is a multi-array of real entries $a_{i_1 \dots i_m}$, where $i_j \in \{1, 2, \dots, n\}$ for $j \in \{1, 2, \dots, m\}$. Denote the set of all real m th order n -dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension n^m . Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$. If the entries $a_{i_1 \dots i_m}$ are invariant under any permutation of their indices, then \mathcal{A} is called a symmetric tensor. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in \mathbb{R}^n with its i th component as

$$(\mathcal{A}\mathbf{x}^{m-1})_i := \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m}$$

for $i \in \{1, 2, \dots, n\}$ ([13]).

For m -order finite dimensional tensor, various structured tensors were studied well. For more details, M-tensors see Zhang, Qi and Zhou [14] and Ding, Qi and Wei [15]; P-(B-)tensors see Song and Qi [16], Qi and Song [17]; copositive tensors see Song and Qi [18]; Cauchy tensor see Chen and Qi [19]; the applications in nonlinear complementarity problem see Song and Qi [20], Che, Qi, Wei [21], Song and Yu [22], Luo, Qi and Xiu [23], Gowda, Luo, Qi and Xiu [24], Bai, Huang and Wang [25], Wang, Huang and Bai [26], Ding, Luo and Qi [27], Suo and Wang [28], Song and Qi [29], Ling, He, Qi [30, 31], Chen, Yang, Ye [32].

In this paper, we show that an m -order infinite dimensional Hilbert tensor defines an $(m-1)$ -homogeneous operator on the spaces of analytic functions (Hardy spaces H^p ($p > m-1$) and Bergman spaces A^p ($p > 2(m-1)$)),

$$\mathcal{H}(f)(z) = \sum_{k=0}^{+\infty} \left(\sum_{i_2, i_3, \dots, i_m=0}^{+\infty} \frac{a_{i_2} a_{i_3} \cdots a_{i_m}}{k + i_2 + i_3 + \cdots + i_m + 1} \right) z^k \quad (1.5)$$

for all analytic functions $f(z) = \sum_{k=0}^{+\infty} a_k z^k$. The upper bound of Hilbert tensor operator $\mathcal{H}(f)$ is found on Bergman spaces A^p ($p > 2(m-1)$) with the help of the proof technique of Diamantopoulos [9]. So two positively homogeneous operators may be defined on Bergman spaces A^p by the formula

$$T_{\mathcal{H}}(f)(z) := \begin{cases} \|f\|_{A^{p(m-1)}}^{2-m} \mathcal{H}(f)(z), & f \neq 0 \\ 0, & f = 0 \end{cases} \quad \text{and} \quad F_{\mathcal{H}}(f)(z) := (\mathcal{H}(f)(z))^{\frac{1}{m-1}} \quad (m \text{ is even}), \quad (1.6)$$

where $T_{\mathcal{H}} : A^{p(m-1)} \rightarrow A^p$ and $F_{\mathcal{H}} : A^p \rightarrow A^p$. We obtain the upper bound of operator norm $\|T_{\mathcal{H}}\|$ and $\|F_{\mathcal{H}}\|$. In particular, when $p = 4(m-1)$,

$$\|T_{\mathcal{H}}\| \leq \pi \text{ and } \|F_{\mathcal{H}}\| \leq \pi^{\frac{1}{m-1}}. \quad (1.7)$$

The paper is organized as follows: In Section 2, we will give some basic definitions and facts, which will be used to the proof of main results. In Section 3, we first study the definition of Hilbert tensor operator and give the corresponding proof to show that such an operator is well-defined. We prove the intergral form of Hilbert tensor operator. Secondly, the boundedness of Hilbert tensor operator is proved on Bergman spaces A^p ($p > 2(m-1)$) by means of its intergral form. Finally, we define two positively homogeneous operators induced by m -order infinite dimensional Hilbert tensor and prove the upper boundedness of their operator norm.

2 Preliminaries and basic facts

In this section, we will collect some basic definitions and facts, which will be used later on. Throughout this paper, let \mathbb{C} be the complex plane, and let

$$\mathbb{B} := \{z \in \mathbb{C} : \|z\| < 1\}$$

be the open unit disk in \mathbb{C} . Likewise, we write \mathbb{R} for the real line. The normalized Lebesgue measure on \mathbb{B} will be denoted by $d\mu$. Obviously,

$$d\mu(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$$

for $z = x + yi = re^{i\theta}$. For $0 < p < +\infty$, the Bergman space A^p is a space of all analytic functions f in \mathbb{B} with

$$\|f\|_{A^p} = \left(\int_{\mathbb{B}} |f(z)|^p d\mu(z) \right)^{\frac{1}{p}} < +\infty. \quad (2.1)$$

The Hardy space H^p is a space of all analytic functions f in \mathbb{B} with

$$\|f\|_{H^p} = \sup_{r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty. \quad (2.2)$$

It is well-known that both Hardy space H^p and Bergman space A^p are a Banach space for $1 \leq p$, and $H^p \subset A^p$, and both H^p and A^p are embeded as a closed subspaces in Lebesgue space $L^p(\mathbb{B})$, and $H^q \subset H^p$, $A^q \subset A^p$ for $q \leq p$ (for more details, see [33, 34]).

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach space, and let $T : K \subset X \rightarrow Y$ be an operator and let $r \in \mathbb{R}$. T is called

- (i) *r-homogeneous* if $T(tx) = t^r Tx$ for each $t \in \mathbb{C}$ and all $x \in K$;
- (ii) *positively homogeneous* if $T(tx) = tTx$ for each $t > 0$ and all $x \in K$;
- (iii) *bounded* if there is a real number $M > 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X, \text{ for all } x \in K.$$

The gamma function $\Gamma(z)$ is defined by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \quad (2.3)$$

whenever the complex variable z has a positive real part, i.e., $\Re(z) > 0$. The beta function $\beta(u, v)$ is defined by the formula

$$\beta(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt, \quad \Re(u) > 0, \quad \Re(v) > 0. \quad (2.4)$$

The formula relating the beta function to the gamma function is the following:

$$\beta(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \quad (2.5)$$

Furthermore, the gamma function has the following properties ([35]):

- (i) $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$;
- (ii) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ for non-integral complex numbers z .
- (iii) The duplication formula: $\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z} \sqrt{\pi} \Gamma(2z)$, i.e.,

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z} \frac{\Gamma(z)\Gamma(\frac{1}{2})}{\Gamma(z + \frac{1}{2})}. \quad (2.6)$$

Lemma 2.1. ([33, Page 36, Lemma]) If $f \in H^p$ and $0 < p < +\infty$, then

$$|f(z)| \leq \left(\frac{2}{1-|z|} \right)^{\frac{1}{p}} \|f\|_{H^p}. \quad (2.7)$$

Lemma 2.2. ([36, Page 755, Corollary]) If $f \in A^p$ and $0 < p < +\infty$, then

$$|f(z)| \leq \left(\frac{1}{1-|z|^2} \right)^{\frac{2}{p}} \|f\|_{A^p}. \quad (2.8)$$

3 Hilbert tensor operators

3.1 Integral form of Hilbert tensor operator

Lemma 3.1. Let \mathcal{H} be an m -order infinite dimensional Hilbert tensor, and let $f(z) = \sum_{k=0}^{+\infty} a_k z^k \in L^{m-1}(\mathbb{B})$. Then

$$\mathcal{H}(f)(z) = \sum_{k=0}^{+\infty} \left(\sum_{i_2, i_3, \dots, i_m=0}^{+\infty} \frac{a_{i_2} a_{i_3} \cdots a_{i_m}}{k + i_2 + i_3 + \cdots + i_m + 1} \right) z^k \quad (3.1)$$

is a well-defined analytic function on the unit disc \mathbb{B} . Furthermore, $\mathcal{H}(f)(z)$ is a well-defined on Hardy space H^p or on Bergman space A^p ($m-1 < p < +\infty$).

Proof. Let $f_l(z) = \sum_{k=0}^l a_k z^k$ for all positive integer l . Obviously, $\lim_{l \rightarrow \infty} f_l(z) = f(z)$, and so, $\lim_{l \rightarrow \infty} (f_l(z))^{m-1} = (f(z))^{m-1}$. Thus for each $z \in \mathbb{B}$, there is a positive integer N such that $|f_l(z)|^{m-1} \leq |f(z)|^{m-1} + 1$ for all positive integer $l > N$. So for all positive integer $l > N$, we have

$$\begin{aligned}
\left| \sum_{i_2, i_3, \dots, i_m=0}^l \frac{a_{i_2} a_{i_3} \cdots a_{i_m}}{i_1 + i_2 + \cdots + i_m + 1} \right| &= \left| \sum_{i_2, i_3, \dots, i_m=0}^l a_{i_2} a_{i_3} \cdots a_{i_m} \int_0^1 s^{i_1 + \cdots + i_m} ds \right| \\
&= \left| \int_0^1 \left(\sum_{i_2, i_3, \dots, i_m=0}^l a_{i_2} a_{i_3} \cdots a_{i_m} s^{i_2 + \cdots + i_m} \right) s^{i_1} ds \right| \\
&= \left| \int_0^1 \left(\sum_{i=0}^l a_i s^i \right)^{m-1} s^{i_1} ds \right| \\
&= \left| \int_0^1 (f_l(s))^{m-1} s^{i_1} ds \right| \\
&\leq \int_0^1 |f_l(s)|^{m-1} |s|^{i_1} ds \leq \int_0^1 |f_l(s)|^{m-1} ds \\
&\leq \int_0^1 (|f(s)|^{m-1} + 1) ds < +\infty \quad (\text{since } f \in L^{m-1}(\mathbb{B})).
\end{aligned}$$

Then,

$$\left| \sum_{i_2, i_3, \dots, i_m=0}^{+\infty} \frac{a_{i_2} a_{i_3} \cdots a_{i_m}}{i_1 + i_2 + \cdots + i_m + 1} \right| < +\infty,$$

and hence the convergence radius of the following power series, denoted by $\mathcal{H}(f)(z)$,

$$\mathcal{H}(f)(z) = \sum_{k=0}^{+\infty} \left(\sum_{i_2, i_3, \dots, i_m=0}^{+\infty} \frac{a_{i_2} a_{i_3} \cdots a_{i_m}}{k + i_2 + i_3 + \cdots + i_m + 1} \right) z^k$$

is greater than or equal to 1. So $\mathcal{H}(f)(z)$ is a well-defined analytic function on the unit disc \mathbb{B} . The desired conclusions follow. \square

Lemma 3.2. *Let*

$$\mathcal{G}(f)(z) = \int_0^1 \frac{(f(s))^{m-1}}{1 - zs} ds \quad (m \geq 2) \tag{3.2}$$

for $z \in \mathbb{B}$. *The operator* $\mathcal{G}(f)(z)$ *is well-defined on Hardy space* H^p $(m-1 < p < +\infty)$ *or on Bergman space* A^p $(2(m-1) < p < +\infty)$.

Proof. (1) For $f \in H^p$, from Lemma 2.1 and the fact that

$$\frac{1}{|1 - zs|} \leq \frac{1}{1 - |z||s|} \leq \frac{1}{1 - |z|},$$

it follows that

$$\begin{aligned}
|\mathcal{G}(f)(z)| &\leq \int_0^1 \frac{(|f(s)|)^{m-1}}{|1-zs|} ds \\
&\leq \int_0^1 \frac{\left(\left(\frac{2}{1-s}\right)^{\frac{1}{p}} \|f\|_{H^p}\right)^{m-1}}{1-|z|} ds \\
&= \frac{2^{\frac{m-1}{p}} \|f\|_{H^p}^{m-1}}{1-|z|} \int_0^1 \frac{1}{(1-s)^{\frac{m-1}{p}}} ds < +\infty \quad \left(\frac{m-1}{p} < 1\right)
\end{aligned}$$

since the integral $\int_0^1 \frac{1}{(1-s)^r} ds$ converges for $r < 1$.

(2) For $f \in A^p$, it follows from Lemma 2.2 that

$$\begin{aligned}
|\mathcal{G}(f)(z)| &\leq \int_0^1 \frac{\left(\left(\frac{1}{1-s}\right)^{\frac{2}{p}} \|f\|_{A^p}\right)^{m-1}}{1-|z|} ds \\
&= \frac{\|f\|_{A^p}^{m-1}}{1-|z|} \int_0^1 \frac{1}{(1-s)^{\frac{2(m-1)}{p}}} ds < +\infty \quad \left(\frac{2(m-1)}{p} < 1\right).
\end{aligned}$$

The desired conclusions follow. \square

Lemma 3.3. Let \mathcal{H} be an m -order infinite dimensional Hilbert tensor, and let $f \in H^p$ ($m-1 < p < +\infty$) or $f \in A^p$ ($2(m-1) < p < +\infty$). Then for each $z \in \mathbb{B}$,

$$(i) \quad \mathcal{H}(f)(z) = \mathcal{G}(f)(z) = \int_0^1 \frac{(f(s))^{m-1}}{1-zs} ds;$$

$$(ii) \quad \mathcal{H}(f)(z) = \mathcal{G}(f)(z) = \int_0^1 \left(f\left(\frac{s}{(s-1)z+1}\right)\right)^{m-1} \frac{1}{(s-1)z+1} ds.$$

Proof. (i) Let $f_l(z) = \sum_{k=0}^l a_k z^k$. Obviously, $\lim_{l \rightarrow \infty} f_l(z) = f(z)$, and hence, $\lim_{l \rightarrow \infty} (f_l(z))^{m-1} = (f(z))^{m-1}$. Now we may define a polynomial

$$\mathcal{H}(f_l)(z) = \sum_{k=0}^{+\infty} \left(\sum_{i_2, i_3, \dots, i_m=0}^l \frac{a_{i_2} a_{i_3} \cdots a_{i_m}}{k + i_2 + i_3 + \cdots + i_m + 1} \right) z^k.$$

Then we have

$$\begin{aligned}
\mathcal{H}(f_l)(z) &= \sum_{k=0}^{+\infty} z^k \sum_{i_2, i_3, \dots, i_m=0}^l a_{i_2} a_{i_3} \cdots a_{i_m} \int_0^1 s^{k+i_2+\dots+i_m} ds \\
&= \sum_{k=0}^{+\infty} z^k \int_0^1 \left(\sum_{i_2, i_3, \dots, i_m=0}^l a_{i_2} a_{i_3} \cdots a_{i_m} s^{i_2+\dots+i_m} \right) s^k ds \\
&= \sum_{k=0}^{+\infty} z^k \int_0^1 \left(\sum_{i=0}^l a_i s^i \right)^{m-1} s^k ds \\
&= \sum_{k=0}^{+\infty} \int_0^1 (f_l(s))^{m-1} (zs)^k ds \\
&= \int_0^1 (f_l(s))^{m-1} \left(\sum_{k=0}^{+\infty} (zs)^k \right) ds \\
&= \int_0^1 \frac{(f_l(s))^{m-1}}{1-zs} ds.
\end{aligned}$$

For $z \in \mathbb{B}$ and $p > m-1$, it is obvious that the fact that $f(z) \in H^p$ implies that $(f(z))^{m-1} \in H^{\frac{p}{m-1}}$, and hence, $(f(z))^{m-1} - (f_l(z))^{m-1} \in H^{\frac{p}{m-1}}$. Furthermore, from Lemma 2.1, it follows that

$$\begin{aligned}
|\mathcal{H}(f_l)(z) - \mathcal{G}(f)(z)| &= \left| \int_0^r \frac{(f_l(s))^{m-1} - (f(s))^{m-1}}{1-zs} ds \right| \\
&\leq \int_0^1 \frac{|(f_l(s))^{m-1} - (f(s))^{m-1}|}{|1-zs|} ds \\
&\leq \int_0^1 \frac{\left(\frac{2}{1-s}\right)^{\frac{m-1}{p}} \|f_l^{m-1} - f^{m-1}\|_{H^{\frac{p}{m-1}}}}{1-|z|} ds \\
&= \left(\frac{2^{\frac{m-1}{p}}}{1-|z|} \int_0^1 \frac{1}{(1-s)^{\frac{m-1}{p}}} ds \right) \|f_l^{m-1} - f^{m-1}\|_{H^{\frac{p}{m-1}}}.
\end{aligned}$$

Therefore, for each $z \in \mathbb{B}$,

$$\lim_{l \rightarrow \infty} \mathcal{H}(f_l)(z) = \sum_{k=0}^{+\infty} \left(\sum_{i_2, i_3, \dots, i_m=0}^{+\infty} \frac{a_{i_2} a_{i_3} \cdots a_{i_m}}{k + i_2 + i_3 + \cdots + i_m + 1} \right) z^k = \mathcal{G}(f)(z).$$

Then $\mathcal{G}(f)(z)$ defines an analytic function $\mathcal{H}(f)(z) = \lim_{l \rightarrow \infty} \mathcal{H}(f_l)(z)$. That is,

$$\mathcal{H}(f)(z) = \mathcal{G}(f)(z) = \int_0^1 \frac{(f(s))^{m-1}}{1-zs} ds$$

for each $f \in H^p$ ($m-1 < p < +\infty$).

Similarly, for $f \in A^p$, it follows from Lemma 2.2 that

$$\begin{aligned}
|\mathcal{H}(f_l)(z) - \mathcal{G}(f)(z)| &\leq \int_0^1 \frac{|(f_l(s))^{m-1} - (f(s))^{m-1}|}{|1 - zs|} ds \\
&\leq \int_0^1 \frac{\left(\frac{1}{1-s}\right)^{\frac{2(m-1)}{p}} \|f_l^{m-1} - f^{m-1}\|_{A^{\frac{p}{m-1}}}}{1 - |z|} ds \\
&= \left(\frac{1}{1 - |z|} \int_0^1 \frac{1}{(1-s)^{\frac{2(m-1)}{p}}} ds \right) \|f_l^{m-1} - f^{m-1}\|_{A^{\frac{p}{m-1}}},
\end{aligned}$$

and so, $\mathcal{H}(f)(z) = \mathcal{G}(f)(z)$ for every $f \in A^p$ ($2(m-1) < p < +\infty$).

(ii) Given $f \in H^p$ ($m-1 < p < +\infty$) or $f \in A^p$ ($2(m-1) < p < +\infty$), the integral $\mathcal{G}(f)(z)$ is independent of the path of integration. Then for $z \in \mathbb{B}$, we may choose the path of integration

$$s(t) = \frac{t}{(t-1)z + 1}, 0 \leq t \leq 1,$$

and hence

$$s'(t) = \frac{ds(t)}{dt} = \frac{((t-1)z + 1) - tz}{((t-1)z + 1)^2} = \frac{1 - z}{((t-1)z + 1)^2}.$$

So we have

$$\begin{aligned}
\mathcal{G}(f)(z) &= \int_0^1 \frac{\left(f\left(\frac{t}{(t-1)z + 1}\right)\right)^{m-1}}{1 - z \frac{t}{(t-1)z + 1}} s'(t) dt \\
&= \int_0^1 \left(f\left(\frac{t}{(t-1)z + 1}\right)\right)^{m-1} \frac{(t-1)z + 1}{(t-1)z + 1 - zt} \frac{1 - z}{((t-1)z + 1)^2} dt \\
&= \int_0^1 \left(f\left(\frac{t}{(t-1)z + 1}\right)\right)^{m-1} \frac{1}{(t-1)z + 1} dt.
\end{aligned}$$

The desired conclusion follows. \square

3.2 Boundedness of Hilbert tensor operator

Theorem 3.1. *Let \mathcal{H} be an m -order infinite dimensional Hilbert tensor, and let $\mathcal{H}(f)$ be as in Lemma 3.1. Then \mathcal{H} is bounded and $(m-1)$ -homogeneous on Bergmans space $A^{p(m-1)}$ for $2 < p < +\infty$, and satisfies the following:*

(i) *If $4 \leq p < +\infty$ and $f \in A^{p(m-1)}$, then*

$$\|\mathcal{H}(f)\|_{A^p} \leq \frac{\pi}{\sin(\frac{2\pi}{p})} \|f\|_{A^{p(m-1)}}^{m-1}. \quad (3.3)$$

(ii) *If $2 < p \leq 4$ and $f \in A^{p(m-1)}$, then*

$$\|\mathcal{H}(f)\|_{A^p} \leq M \|f\|_{A^{p(m-1)}}^{m-1}, \quad (3.4)$$

where $M = 4^{\frac{4}{p}-1} \sqrt{\pi}^{\frac{\Gamma(1-\frac{2}{p})}{\Gamma(\frac{3}{2}-\frac{2}{p})}}.$

Proof. Let

$$\varphi(t, z) = \frac{t}{(t-1)z+1} \text{ and } \psi(t, z) = \frac{1}{(t-1)z+1}$$

for all $z \in \mathbb{B}$ and all real number t with $0 < t < 1$. Then

$$\frac{\partial \varphi(t, z)}{\partial z} = \frac{-t(t-1)}{((t-1)z+1)^2} = t(1-t)(\psi(t, z))^2.$$

Let $\mathcal{T}_t(f)(z) = \psi(t, z) (f(\varphi(t, z)))^{m-1}$. Then for each $t \in (0, 1)$, we have

$$\begin{aligned} \|\mathcal{T}_t(f)\|_{A^p}^p &= \int_{\mathbb{B}} |\psi(t, z)|^p |(f(\varphi(t, z)))^{m-1}|^p d\mu(z) \\ &= \int_{\mathbb{B}} |\psi(t, z)|^{p-4} |f(\varphi(t, z))|^{p(m-1)} |\psi(t, z)|^4 d\mu(z) \\ &= \int_{\mathbb{B}} |\psi(t, z)|^{p-4} |f(\varphi(t, z))|^{p(m-1)} \frac{1}{t^2(1-t)^2} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z) \\ &= \frac{1}{t^2(1-t)^2} \int_{\mathbb{B}} |\psi(t, z)|^{p-4} |f(\varphi(t, z))|^{p(m-1)} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z). \end{aligned}$$

(i) For $+\infty > p \geq 4$ and each $t \in (0, 1)$, we have

$$\psi(t, z) = \frac{\varphi(t, z)}{t},$$

$$|\varphi(t, z)| = \frac{t}{|(t-1)z+1|} \leq \frac{t}{1-|t-1||z|} \leq \frac{t}{1-(1-t)} = 1$$

and furthermore,

$$\begin{aligned} \|\mathcal{T}_t(f)\|_{A^p}^p &= \frac{1}{t^2(1-t)^2} \int_{\mathbb{B}} \left| \frac{\varphi(t, z)}{t} \right|^{p-4} |f(\varphi(t, z))|^{p(m-1)} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z) \\ &= \frac{1}{t^{p-2}(1-t)^2} \int_{\mathbb{B}} |\varphi(t, z)|^{p-4} |f(\varphi(t, z))|^{p(m-1)} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z) \\ &\leq \frac{1}{t^{p-2}(1-t)^2} \int_{\mathbb{B}} |f(\varphi(t, z))|^{p(m-1)} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z) \\ &= \frac{1}{t^{p-2}(1-t)^2} \int_{\mathbb{D}} |f(y)|^{p(m-1)} d\mu(y), \end{aligned}$$

where $y = \varphi(t, z)$, $\mathbb{D} = \{y = \varphi(t, z); z \in \mathbb{B}\}$ and $d\mu(y) = \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z)$. Therefore, we have

$$\begin{aligned} \|\mathcal{T}_t(f)\|_{A^p} &\leq \frac{1}{t^{1-\frac{2}{p}}(1-t)^{\frac{2}{p}}} \left(\int_{\mathbb{D}} |f(y)|^{p(m-1)} d\mu(y) \right)^{\frac{1}{p}} \\ &= \frac{1}{t^{1-\frac{2}{p}}(1-t)^{\frac{2}{p}}} \left(\left(\int_{\mathbb{D}} |f(y)|^{p(m-1)} d\mu(y) \right)^{\frac{1}{p(m-1)}} \right)^{m-1} \\ &= t^{\frac{2}{p}-1} (1-t)^{-\frac{2}{p}} \|f\|_{A^{p(m-1)}}^{m-1}. \end{aligned} \tag{3.5}$$

From the equality

$$\mathcal{H}(f)(z) = \int_0^1 \psi(t, z) (f(\varphi(t, z)))^{m-1} dt = \int_0^1 \mathcal{T}_t(f)(z) dt$$

and the Minkowski's integral inequality, it follows that

$$\begin{aligned} \|\mathcal{H}(f)\|_{A^p} &= \left(\int_{\mathbb{B}} |\mathcal{H}(f)(z)|^p d\mu(z) \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{B}} \left| \int_0^1 \mathcal{T}_t(f)(z) dt \right|^p d\mu(z) \right)^{\frac{1}{p}} \\ &\leq \int_0^1 \left(\int_{\mathbb{B}} |\mathcal{T}_t(f)(z)|^p d\mu(z) \right)^{\frac{1}{p}} dt \\ &= \int_0^1 \|\mathcal{T}_t(f)\|_{A^p} dt, \end{aligned} \tag{3.6}$$

and hence, using (3.5), we have

$$\begin{aligned} \|\mathcal{H}(f)\|_{A^p} &\leq \left(\int_0^1 t^{\frac{2}{p}-1} (1-t)^{(1-\frac{2}{p})-1} dt \right) \|f\|_{A^{p(m-1)}}^{m-1} \\ &= \beta\left(\frac{2}{p}, 1 - \frac{2}{p}\right) \|f\|_{A^{p(m-1)}}^{m-1} \\ &= \frac{\Gamma(\frac{2}{p})\Gamma(1 - \frac{2}{p})}{\Gamma(\frac{2}{p} + 1 - \frac{2}{p})} \|f\|_{A^{p(m-1)}}^{m-1} \\ &= \frac{\pi}{\sin(\frac{2\pi}{p})} \|f\|_{A^{p(m-1)}}^{m-1}. \end{aligned}$$

(ii) For $2 < p \leq 4$ and each $t \in (0, 1)$, we also have

$$|(\psi(t, z))^{-1}| = |(t-1)z + 1| \leq 2,$$

and so,

$$\begin{aligned} \|\mathcal{T}_t(f)\|_{A^p}^p &= \frac{1}{t^2(1-t)^2} \int_{\mathbb{B}} |\psi(t, z)|^{p-4} |f(\varphi(t, z))|^{p(m-1)} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z) \\ &= \frac{1}{t^2(1-t)^2} \int_{\mathbb{B}} |(\psi(t, z))^{-1}|^{4-p} |f(\varphi(t, z))|^{p(m-1)} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z) \\ &\leq \frac{2^{4-p}}{t^2(1-t)^2} \int_{\mathbb{B}} |f(\varphi(t, z))|^{p(m-1)} \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z) \\ &= \frac{2^{4-p}}{t^2(1-t)^2} \int_{\mathbb{D}} |f(y)|^{p(m-1)} d\mu(y), \end{aligned}$$

where $y = \varphi(t, z)$, $\mathbb{D} = \{y = \varphi(t, z); z \in \mathbb{B}\}$ and $d\mu(y) = \left| \frac{\partial \varphi(t, z)}{\partial z} \right|^2 d\mu(z)$. Therefore, we have

$$\begin{aligned} \|\mathcal{T}_t(f)\|_{A^p} &\leq \frac{2^{\frac{4}{p}-1}}{t^{\frac{2}{p}}(1-t)^{\frac{2}{p}}} \left(\left(\int_{\mathbb{D}} |f(y)|^{p(m-1)} d\mu(y) \right)^{\frac{1}{p(m-1)}} \right)^{m-1} \\ &= 2^{\frac{4}{p}-1} t^{-\frac{2}{p}} (1-t)^{-\frac{2}{p}} \|f\|_{A^{p(m-1)}}^{m-1}. \end{aligned} \quad (3.7)$$

From (3.6), (3.7) and the duplication formula (2.6) of the gamma function $\Gamma(\cdot)$, it follows that

$$\begin{aligned} \|\mathcal{H}(f)\|_{A^p} &\leq \int_0^1 \|\mathcal{T}_t(z)\|_{A^p} dt \\ &\leq 2^{\frac{4}{p}-1} \left(\int_0^1 t^{(1-\frac{2}{p})-1} (1-t)^{(1-\frac{2}{p})-1} dt \right) \|f\|_{A^{p(m-1)}}^{m-1} \\ &= 2^{\frac{4}{p}-1} \beta\left(1 - \frac{2}{p}, 1 - \frac{2}{p}\right) \|f\|_{A^{p(m-1)}}^{m-1} \\ &= 2^{\frac{4}{p}-1} \frac{\Gamma(1 - \frac{2}{p}) \Gamma(1 - \frac{2}{p})}{\Gamma(2 - \frac{4}{p})} \|f\|_{A^{p(m-1)}}^{m-1} \\ &= 2^{\frac{4}{p}-1} \left(2^{1-2(1-\frac{2}{p})} \sqrt{\pi} \frac{\Gamma(1 - \frac{2}{p})}{\Gamma(1 - \frac{2}{p} + \frac{1}{2})} \right) \|f\|_{A^{p(m-1)}}^{m-1} \\ &= 4^{\frac{4}{p}-1} \sqrt{\pi} \frac{\Gamma(1 - \frac{2}{p})}{\Gamma(\frac{3}{2} - \frac{2}{p})} \|f\|_{A^{p(m-1)}}^{m-1}. \end{aligned}$$

The desired conclusions follow. \square

Define an operator $T_{\mathcal{H}} : A^{p(m-1)} \rightarrow A^p$ by the formula

$$T_{\mathcal{H}}(f)(z) := \begin{cases} \|f\|_{A^{p(m-1)}}^{2-m} \mathcal{H}(f)(z), & f \neq 0 \\ 0, & f = 0. \end{cases} \quad (3.8)$$

When m is even, define another operator $F_{\mathcal{H}} : A^p \rightarrow A^p$ by the formula

$$F_{\mathcal{H}}(f)(z) := (\mathcal{H}(f)(z))^{\frac{1}{m-1}}. \quad (3.9)$$

Clearly, both $F_{\mathcal{H}}$ and $T_{\mathcal{H}}$ are bounded and positively homogeneous by Theorem 3.1. So we may define the following the operator norms ([38]):

$$\|T_{\mathcal{H}}\| = \sup_{\|f\|_{A^{p(m-1)}}=1} \|T_{\mathcal{H}}(f)\|_{A^p} \text{ and } \|F_{\mathcal{H}}\| = \sup_{\|f\|_{A^p}=1} \|F_{\mathcal{H}}(f)\|_{A^p}. \quad (3.10)$$

The following upper bounds and properties of the operator norm may be established.

Theorem 3.2. *Let \mathcal{H} be an m -order infinite dimensional Hilbert tensor, and let $\mathcal{H}(f)$ be as in Lemma 3.1. Then $T_{\mathcal{H}}$ is a bounded and positively homogeneous operator from Bergmans space $A^{p(m-1)}$ to A^p for $2 < p < +\infty$, and its norm satisfies the following:*

(i) If $4 \leq p < +\infty$, then

$$\|T_{\mathcal{H}}\| \leq \frac{\pi}{\sin(\frac{2\pi}{p})}. \quad (3.11)$$

(ii) If $2 < p \leq 4$, then

$$\|T_{\mathcal{H}}\| \leq 4^{\frac{4}{p}-1} \sqrt{\pi} \frac{\Gamma(1 - \frac{2}{p})}{\Gamma(\frac{3}{2} - \frac{2}{p})}. \quad (3.12)$$

Proof. It follows the definition (3.8) of the operator $T_{\mathcal{H}}$ that

$$\|T_{\mathcal{H}}(f)\|_{A^p} = \| \|f\|_{A^{p(m-1)}}^{2-m} \mathcal{H}(f) \|_{A^p} = \|f\|_{A^{p(m-1)}}^{2-m} \|\mathcal{H}(f)\|_{A^p}.$$

Then an application of Theorem 3.1 yields the following:

(i) For $4 \leq p < +\infty$,

$$\|T_{\mathcal{H}}(f)\|_{A^p} \leq \|f\|_{A^{p(m-1)}}^{2-m} \left(\frac{\pi}{\sin(\frac{2\pi}{p})} \|f\|_{A^{p(m-1)}}^{m-1} \right) = \frac{\pi}{\sin(\frac{2\pi}{p})} \|f\|_{A^{p(m-1)}}.$$

(ii) For $2 < p \leq 4$,

$$\begin{aligned} \|T_{\mathcal{H}}(f)\|_{A^p} &\leq \|f\|_{A^{p(m-1)}}^{2-m} \left(4^{\frac{4}{p}-1} \sqrt{\pi} \frac{\Gamma(1 - \frac{2}{p})}{\Gamma(\frac{3}{2} - \frac{2}{p})} \|f\|_{A^{p(m-1)}}^{m-1} \right) \\ &= 4^{\frac{4}{p}-1} \sqrt{\pi} \frac{\Gamma(1 - \frac{2}{p})}{\Gamma(\frac{3}{2} - \frac{2}{p})} \|f\|_{A^{p(m-1)}}. \end{aligned}$$

So the desired conclusions directly follow from the definition of operator norm (3.10). \square

Theorem 3.3. Let \mathcal{H} be an m -order infinite dimensional Hilbert tensor, and let $\mathcal{H}(f)$ be as in Lemma 3.1. Then $F_{\mathcal{H}}$ is a bounded and positively homogeneous operator from Bergmans space A^p to A^p for $2(m-1) < p < +\infty$, and its norm satisfies the following:

(i) If $4(m-1) \leq p < +\infty$, then

$$\|F_{\mathcal{H}}\| \leq \left(\frac{\pi}{\sin(\frac{2(m-1)\pi}{p})} \right)^{\frac{1}{m-1}}. \quad (3.13)$$

(ii) If $2(m-1) < p \leq 4(m-1)$, then

$$\|F_{\mathcal{H}}\| \leq 4^{\frac{4}{p}} \left(\frac{\sqrt{\pi} \Gamma(1 - \frac{2(m-1)}{p})}{4 \Gamma(\frac{3}{2} - \frac{2(m-1)}{p})} \right)^{\frac{1}{m-1}}. \quad (3.14)$$

Proof. It follows the definition (3.9) of the operator $F_{\mathcal{H}}$ and the Minkowski's integral inequality that

$$\begin{aligned}
\|F_{\mathcal{H}}(f)\|_{A^p} &= \left(\int_{\mathbb{B}} \left| (\mathcal{H}(f)(z))^{\frac{1}{m-1}} \right|^p d\mu(z) \right)^{\frac{1}{p}} \\
&= \left(\left(\int_{\mathbb{B}} |\mathcal{H}(f)(z)|^{\frac{p}{m-1}} d\mu(z) \right)^{\frac{m-1}{p}} \right)^{\frac{1}{m-1}} \\
&= \left(\left(\int_{\mathbb{B}} \left| \int_0^1 \mathcal{T}_t(f)(z) dt \right|^{\frac{p}{m-1}} d\mu(z) \right)^{\frac{m-1}{p}} \right)^{\frac{1}{m-1}} \\
&\leq \left(\int_0^1 \left(\int_{\mathbb{B}} |\mathcal{T}_t(f)(z)|^{\frac{p}{m-1}} p d\mu(z) \right)^{\frac{m-1}{p}} dt \right)^{\frac{1}{m-1}} \\
&= \left(\int_0^1 \|\mathcal{T}_t(f)\|_{A^{\frac{p}{m-1}}}^{\frac{p}{m-1}} dt \right)^{\frac{1}{m-1}}, \tag{3.15}
\end{aligned}$$

Using the proof technique of Theorem 3.1 (p is replaced by $\frac{p}{m-1}$), the followings may be proved easily:

(i) For $4 \leq \frac{p}{m-1} < +\infty$,

$$\|\mathcal{T}_t(f)\|_{A^{\frac{p}{m-1}}} \leq t^{\frac{2(m-1)}{p}-1} (1-t)^{-\frac{2(m-1)}{p}} \|f\|_{A^p}^{m-1},$$

and hence,

$$\|F_{\mathcal{H}}(f)\|_{A^p} \leq \left(\frac{\pi}{\sin(\frac{2(m-1)\pi}{p})} \|f\|_{A^p}^{m-1} \right)^{\frac{1}{m-1}} = \left(\frac{\pi}{\sin(\frac{2(m-1)\pi}{p})} \right)^{\frac{1}{m-1}} \|f\|_{A^p}.$$

(ii) For $2 < \frac{p}{m-1} \leq 4$,

$$\|\mathcal{T}_t(f)\|_{A^{\frac{p}{m-1}}} \leq 2^{\frac{4(m-1)}{p}-1} t^{-\frac{2(m-1)}{p}} (1-t)^{-\frac{2(m-1)}{p}} \|f\|_{A^p}^{m-1},$$

and hence,

$$\begin{aligned}
\|F_{\mathcal{H}}(f)\|_{A^p} &\leq \left(4^{\frac{4(m-1)}{p}-1} \sqrt{\pi} \frac{\Gamma(1 - \frac{2(m-1)}{p})}{\Gamma(\frac{3}{2} - \frac{2(m-1)}{p})} \|f\|_{A^p}^{m-1} \right)^{\frac{1}{m-1}} \\
&= 4^{\frac{4}{p}} \left(\frac{\sqrt{\pi} \Gamma(1 - \frac{2(m-1)}{p})}{4\Gamma(\frac{3}{2} - \frac{2(m-1)}{p})} \right)^{\frac{1}{m-1}} \|f\|_{A^p}.
\end{aligned}$$

So the desired conclusions directly follow from the definition of operator norm (3.10). \square

Let $p = 4$ in Theorem 3.2 ((i) or (ii)) and $p = 4(m - 1)$ in Theorem 3.3((i) or (ii)), respectively. Then the following conclusions are easily obtained.

Corollary 3.4. *Let \mathcal{H} be an m -order infinite dimensional Hilbert tensor, and let $\mathcal{H}(f)$ be as in Lemma 3.1. Then*

(i) $T_{\mathcal{H}} : A^{4(m-1)} \rightarrow A^4$ is a bounded and positively homogeneous operator and

$$\|T_{\mathcal{H}}\| \leq \pi;$$

(ii) $F_{\mathcal{H}} : A^{4(m-1)} \rightarrow A^{4(m-1)}$ is a bounded and positively homogeneous operator and

$$\|F_{\mathcal{H}}\| \leq \pi^{\frac{1}{m-1}}.$$

Remark (i) In this paper, the boundedness of Hilbert tensor operator is obtained on A^p for $p > 2(m - 1)$. For $0 < p \leq 2(m - 1)$, whether or not Hilbert tensor operator is bounded on Bergman space A^p or Hardy space H^p or other space of analytic functions.

(ii) Are the upper bounds of norm of Hilbert tensor operator the best in this paper?

(iii) May the operator norms of $T_{\mathcal{H}}$ and $F_{\mathcal{H}}$ be given the exact value?

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